

A note on the Ramsey number of even wheels versus stars

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Abstract For two graphs G_1 and G_2 , the *Ramsey number* $R(G_1, G_2)$ is the smallest integer N , such that for any graph on N vertices, either G contains G_1 or \overline{G} contains G_2 . Let S_n be a *star* of order n and W_m be a *wheel* of order $m + 1$. In this paper, it is shown that $R(W_n, S_n) \leq 5n/2 - 1$, where $n \geq 6$ is even. It was proven a theorem which implies that $R(W_n, S_n) \geq 5n/2 - 2$, where $n \geq 6$ is even. Therefore we conclude that $R(W_n, S_n) = 5n/2 - 2$ or $5n/2 - 1$, for $n \geq 6$ and even.

Keywords Ramsey number · Star · Wheel · Weakly pancyclic.

MSC: 05C55; 05D10

1 Introduction and Background

Let $G = (V, E)$ denote a finite simple graph on the vertex set V and the edge set E . The subgraph of G *induced* by $S \subseteq V$, $G[S]$, is a graph with vertex set S and two vertices of S are adjacent in $G[S]$ if and only if they are adjacent in G . The complement of a graph G , which is denoted by \overline{G} , is the graph with vertex set $V(G)$ and two vertices in \overline{G} are adjacent if and only if they are not adjacent in G . For a vertex $v \in V(G)$, we denote the set of all neighbors of

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v by $N_G(v)$ (or $N(v)$). The degree of a vertex v in a graph G , denoted by $\deg_G(v)$ (or $\deg(v)$), is the size of the set $N(v)$. A vertex of a connected graph is a *cut-vertex* if its removal produces a disconnected graph.

The graph K_n is the complete graph on n vertices, and C_n is the cycle graph on n vertices. The minimum degree, maximum degree and clique number of G are denoted by $\delta(G)$, $\Delta(G)$ and $\omega(G)$, respectively. The *girth* of graph G , $g(G)$, is the length of shortest cycle. Also, the *circumference* of graph G , is the length of longest cycle in G and denoted by $c(G)$. A graph G of order n is called *Hamiltonian*, *pancyclic* and *weakly pancyclic* if it contains C_n , cycles of every length between 3 and n , and cycles of every length l with $g(G) \leq l \leq c(G)$, respectively. We say that G is a *join* graph if G is the complete union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. In other words, $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$. If G is the join graph of G_1 and G_2 , we shall write $G = G_1 + G_2$. Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2 and edge sets E_1 and E_2 , respectively. A *wheel* W_m is a graph on $m+1$ vertices obtained from C_m by adding one vertex which is called the *hub* and joining each vertex of C_m to the hub with the edges called the *rim* of the wheel. In other words, $W_m = C_m + K_1$. A *star* S_n is the complete bipartite graph $K_{1,n-1}$.

A (proper) coloring is a function $c : V(G) \rightarrow \mathbb{N}$ (where \mathbb{N} is the set of positive integers) such that $c(u) \neq c(v)$ if u and v are adjacent in G . A graph G is *k-colorable* if there exists a coloring of G from a set of k colors. The minimum positive integer k for which G is *k-colorable* is the *chromatic number* of G and is denoted by $\chi(G)$.

For two graphs G_1 and G_2 , the Ramsey number $R(G_1, G_2)$ is the smallest positive integer N such that for every graph G on N vertices, G contains G_1 as a subgraph or the complement of G contains G_2 as a subgraph.

Harary et al. in [2] proved the following lower bound for Ramsey numbers:

$$R(G, H) \geq (\chi(G) - 1) \cdot (l(H) - 1) + 1,$$

where $l(H)$ is the number of vertices in the largest connected component of H .

In this note we consider the Ramsey number for stars versus wheels. There are a lot of results about this subject. The Harary lower bound for $R(W_m, S_n)$ is $3n - 2$ or $2n - 1$, where m is odd or even, respectively.

There are many results about this Ramsey number when m is odd. Chen et al. in [5] proved that if $m \leq n + 1$ and m is odd, then $R(W_m, S_n) = 3n - 2$. Hasmawati et al. in [6] also showed that $R(W_m, S_n) = 3n - 2$, for the case $m \leq 2n - 3$. But, one can see in [7], if $n \geq 2$ and $m \geq 2n - 2$, then $R(W_m, S_n) = n + m - 1$, where m is odd.

Also, one can find many results about $R(W_m, S_n)$ when m is even.

It was shown in [4] that $R(W_4, S_n) = 2n - 1$ if $n \geq 3$ and odd, and $R(W_4, S_n) = 2n + 1$ if $n \geq 4$ and even. Korolova in [1] proved that: $R(W_m, S_n) \geq 2n + 1$ for all $n \geq m \geq 6$ and m even. Also, Chen et al. in [5] showed that $R(W_6, S_n) = 2n + 1$.

It was proven in [8] that $R(W_8, S_n) = 2n + 2$ for $n \geq 6$ and even. Also, it was Shown in [9] that $R(W_8, S_n) = 2n + 1$ for $n \geq 5$ and odd.

Li et al. in [10] indicated two following theorems in which they obtained a new lower bound and showed that for some cases this bound is sharp.

Theorem 1 [10] *If $6 \leq m \leq 2n - 4$ and m is even, then*

$$R(W_m, S_n) \geq \begin{cases} 2n + m/2 - 3 & \text{if } n \text{ is odd and } m/2 \text{ is even} \\ 2n + m/2 - 2 & \text{otherwise.} \end{cases}$$

Theorem 2 [10] *If $n + 1 \leq m \leq 2n - 4$ and m is even, then*

$$R(W_m, S_n) = \begin{cases} 2n + m/2 - 3 & \text{if } n \text{ is odd and } m/2 \text{ is even} \\ 2n + m/2 - 2 & \text{otherwise.} \end{cases}$$

But for some cases, $R(W_m, S_n)$ where m is even, is still open. one of these cases is when $m = n$. It was shown in [1] that $R(W_n, S_n) \leq 3n - 3$ when n is even. In this paper, we want to improve this upper bound and prove that:

Theorem 3 $R(W_n, S_n) \leq 5n/2 - 1$, where $n \geq 6$ is even.

2 Preliminary Lemmas and Theorems

To prove Theorem 3, we need some theorems and lemmas.

Lemma 1 (Brandt et al. [11]). *Every non-bipartite graph G of order n with $\delta(G) \geq (n + 2)/3$ is weakly pancyclic with $g(G) = 3$ or 4.*

Lemma 2 (Dirac [12]). *Let G be a 2-connected graph of order $n \geq 3$ with $\delta(G) = \delta$. Then $c(G) \geq \min\{2\delta, n\}$.*

Theorem 4 (Faudree and Schelp [13], Rosta [14])

$$R(C_n, C_m) = \begin{cases} 2n - 1 & \text{for } 3 \leq m \leq n, \ m \text{ odd } (n, m) \neq (3, 3) \\ n + m/2 - 1 & \text{for } 4 \leq m \leq n, \ m, n \text{ even } (n, m) \neq (4, 4) \\ \max\{n + m/2 - 1, 2m - 1\} & \text{for } 4 \leq m < n, \ m \text{ even and } n \text{ odd.} \end{cases}$$

Lemma 3 [15] *Let G be a bipartite graph of order n (n even) with bipartition (X, Y) and $|X| = |Y| = n/2$. If for all distinct nonadjacent vertices $u \in X$ and $v \in Y$, we have $\deg(u) + \deg(v) > n/2$, then G is Hamiltonian.*

3 Proof of the Theorem 3

From now on, let G be a graph of order $N = 5n/2 - 1$ where $n \geq 6$ and is even, such that neither G contains W_n nor its complement, \overline{G} , contains S_n . Also, for every vertex $t \in V(G)$ consider $H_t = G[N(t)]$. Since \overline{G} has no S_n , $\deg_{\overline{G}}(v) \leq n - 2$. Thus, $\delta(G) \geq 3n/2$. In the middle of the proof, we sometimes interrupt it and have some lemmas.

Let $v_0 \in V(G)$ be an arbitrary vertex. there exists a $k \in \{0, 1, 2, \dots, n -$

2} such that $\deg_G(v_0) = 3n/2 + k$ since $\delta(G) \geq 3n/2$. Thus, the order of $H_{v_0} = G[N(v_0)]$ is $3n/2 + k$. By the second part of the Theorem 4, we have $|V(H_{v_0})| = 3n/2 + k \geq R(C_n, C_s)$, where $s = 2t$, and t is an integer such that $4 \leq 2t \leq n + k + 1$. (Note that in Theorem 3, we have $n \geq 6$, so the case $(n, s) = (4, 4)$ does not occur for $R(C_n, C_s)$ in Theorem 4). Thus, either H_{v_0} contains C_n or \overline{H}_{v_0} contains C_s . But if H_{v_0} contains C_n , then G contains W_n , which is a contradiction. Hence we have the following corollary.

Corollary 1 *Let $v \in V(G)$ and k be an element in the set $\{0, 1, \dots, n-2\}$ such that $|V(H_v)| = 3n/2 + k$. Then \overline{H}_v contains C_{2t} for all integers t such that $4 \leq 2t \leq n + k + 1$.*

Proposition 1 $\omega(\overline{G}) \leq n-2$ and $\omega(G) \leq n-1$.

Proof It is clear that $\omega(\overline{G}) \leq n-1$ since $\Delta(\overline{G}) \leq n-2$. Suppose $\omega(\overline{G}) = n-1$ and $T = \{v_1, \dots, v_{n-1}\}$ is a clique in \overline{G} . For any $v \in V - T$, $N_{\overline{G}}(v) \cap T = \emptyset$ otherwise $\overline{G}[T \cup \{v\}]$ contains S_n . Now consider $v \in V - T$ and k be an element in the set $\{0, 1, \dots, n-2\}$ such that $|V(H_v)| = 3n/2 + k$. The subgraph \overline{H}_v is a disconnected graph with a connected component $\overline{G}[T]$. On the other hand, by Corollary 1, \overline{H}_v contains a cycle C of length $2t$ where $t = \lfloor (n+k+1)/2 \rfloor$. Note that $C \not\subseteq T$, since $2t > n-1$. Thus, $C \subseteq \overline{H}_v - T$. But $\overline{H}_v - T$ has $n/2 + k + 1$ vertices, which is a contradiction. Hence $\omega(\overline{G}) \leq n-2$. For the second part, assume to the contrary, G contains K_n and $H = G[V - K_n]$. Then $|N_G(v) \cap K_n| \geq 2$ for all $v \in V(H)$, otherwise $\deg_{\overline{G}}(v) \geq n-1$, which is a contradiction. If $|N_G(v) \cap K_n| = 2$ for all $v \in V(H)$, then $H = K_{3n/2-1}$ since $\delta(G) \geq 3n/2$. But $K_{3n/2-1}$ contains W_n , a contradiction. So, there is a vertex $u \in V(H)$ such that $|N_G(u) \cap K_n| \geq 3$. But $\{u\} \cup K_n$ contains W_n , which is a contradiction. Thus, $\omega(G) \leq n-1$.

We can divide the proof into some cases and subcases:

Case 1. There is a vertex $v \in V(G)$ for which H_v is bipartite.

Let H_v be a bipartite graph with bipartition (X_v, Y_v) of order $3n/2 + k$ such that $k \in \{0, 1, \dots, n-2\}$. Without loss of generality, suppose that $|X_v| \leq |Y_v|$. Thus, by Proposition 1, we have $n/2 + k + 2 \leq |X_v| \leq 3n/4 + k/2$ and $3n/4 + k/2 \leq |Y_v| \leq n-2$.

Let $|X_v| = n/2 + s$, where s is an integer such that $k+2 \leq s \leq n/4 + k/2$, then $|Y_v| = n + k - s$. Since $\Delta(\overline{G}) \leq n-2$ and $|H_v| = 3n/2 + k$, we conclude $\delta(H_v) \geq n/2 + k + 1$. Let X'_v and Y'_v obtained from X_v and Y_v by deleting s and $n/2 + k - s$ arbitrary vertices, respectively, and let $H'_v = (X'_v, Y'_v)$. Thus, $|X'_v| = |Y'_v| = n/2$ and $\delta(X'_v) \geq s+1$ and $\delta(Y'_v) \geq n/2 + k + 1 - s$ in H'_v . Hence for each two vertices $u_1 \in X'_v$ and $u_2 \in Y'_v$, we have $\deg(u_1) + \deg(u_2) \geq n/2 + k + 2$ and by Lemma 3, H'_v contains C_n . It means that G contains W_n , which is a contradiction.

Case 2. For every vertex $t \in V(G)$, H_t is non-bipartite.

Subcase 2.1. Suppose H_t is disconnected for all $t \in V(G)$.

Let $t \in V(G)$ be an arbitrary vertex and $|V(H_t)| = 3n/2 + k$, where $k \in \{0, 1, 2, \dots, n-2\}$. We show that H_t has exactly two connected components. Suppose to the contrary, H_1, H_2 and H_3 are three connected components of H_t . Since $\delta(H_t) \geq n/2 + k + 1$, we conclude $\delta(H_i) \geq n/2 + k + 1$ for $i = 1, 2, 3$. Hence $|V(H_t)| > 3n/2 + k$, which is a contradiction. Now, let X_t, Y_t be the set of vertices of two components of H_t . Assume that $|X_t| \leq |Y_t|$. We choose two adjacent vertices u and v in Y_t since $\delta(H_t) \geq n/2 + k + 1$. Let $|V(H_u)| = 3n/2 + k'$ and $|V(H_v)| = 3n/2 + k''$, where $k', k'' \in \{0, 1, 2, \dots, n-2\}$. Also, let X_u, Y_u and X_v, Y_v be the set of vertices of two components of H_u and H_v , respectively. Since, H_t and H_u are disconnected, X_u or Y_u is disjoint from X_t and Y_t . To see this, with no loss of generality, suppose that v is contained in Y_u . Thus, $t \in Y_u$ and hence $X_u \cap Y_t = X_u \cap X_t = \emptyset$. Similarly, X_v or Y_v , say X_v , is disjoint from X_t and Y_t . Thus, we have $Y_t \cap X_u = Y_t \cap X_v = X_t \cap X_u = X_t \cap X_v = \emptyset$. Also, $X_u \cap X_v = \emptyset$ otherwise if $l \in X_u \cap X_v$, then l is adjacent to both u and v . But $u \in Y_v$ implies that $l \in Y_v$. It means, $X_v \cap Y_v \neq \emptyset$ which is a contradiction. Thus, $X_u \cap X_v = \emptyset$. Hence $|V(G)| \geq |V(H_t)| + |X_u| + |X_v|$ which means $|V(G)| \geq (3n/2 + k) + (n/2 + k' + 2) + (n/2 + k'' + 2) > 5n/2 - 1$, which is a contradiction.

Subcase 2.2. Suppose H_t is connected for some $t \in V(G)$.

Assume that there exists a vertex $u \in V(G)$ for which H_u is 2-connected and $|V(H_u)| = 3n/2 + k$ for some $k \in \{0, 1, 2, \dots, n-2\}$. Thus, $\delta(H_u) \geq n/2 + k + 1 \geq (3n/2 + k + 2)/3$ and by Lemma 1, H_u is weakly pancyclic with $g(G) = 3$ or 4. Also, by Lemma 2, $c(H_u) \geq \min\{2\delta, 3n/2 + k\}$. Hence $c(H_u) \geq n$ which implies that H_u contains C_n , a contradiction.

Now, assume each connected H_t contains a cut-vertex. Let u be a cut-vertex of H_t and $|V(H_t)| = 3n/2 + k$. We show that $H_t - u$ has exactly two connected components. Suppose to the contrary, H_1, H_2 and H_3 are three connected components of $H_t - u$. Since $\delta(H_t) \geq n/2 + k + 1$, $\delta(H_i) \geq n/2 + k$ for $i = 1, 2, 3$. Hence $|H_t| > 3n/2 + k$, which is a contradiction. Now, let s_1 be a cut-vertex of H_t and X_t, Y_t be the set of vertices of two components of $H_t - s_1$. Assume that $|X_t| \leq |Y_t|$. We choose two adjacent vertices u and v in Y_t since $\delta(H_t) \geq n/2 + k + 1$. Let s_2 and s_3 be the cut-vertices of H_u and H_v , respectively (if any of these cut-vertices didn't exist, for instance s_1 , then the corresponding subgraph, H_t , is disconnected and the procedure is the same as subcase 2.1) and $|V(H_u)| = 3n/2 + k'$ and $|V(H_v)| = 3n/2 + k''$, where $k', k'' \in \{0, 1, 2, \dots, n-2\}$. Also, let X_u, Y_u and X_v, Y_v be the set of vertices of two components of $H_u - s_2$ and $H_v - s_3$, respectively. Since, $H_t - s_1, H_u - s_2$ and $H_v - s_3$ are disconnected, with the same statement of subcase 2.1 and without loss of generality, we have $Y_t \cap X_u = Y_t \cap X_v = X_t \cap X_u = X_t \cap X_v = X_u \cap X_v = \emptyset$. Hence $|V(G)| \geq |V(H_t - s_1)| + |X_u| + |X_v|$ which means $|V(G)| \geq (3n/2 + k - 1) + (n/2 + k' + 1) + (n/2 + k'' + 1) > 5n/2 - 1$, which is a contradiction, and this completes the proof.

Now, by Theorem 1 and 3, the following Corollary is obvious.

Corollary 2 *For $n \geq 6$ and even, we have $R(W_n, S_n) = 5n/2 - 2$ or $5n/2 - 1$.*

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